It is proposed to send out the notice early in December with the names of all signers received up to that time. The meeting will be called at Columbus, Ohio, in connection with the holiday convocation of the American Association for the Advancement of Science. The name of the new society, its precise character and policy, its relation to the AMERICAN MATHEMATICAL MONTHLY, etc., will be questions for full discussion and determination at the organization meeting.

It should be clearly understood that this whole movement is a matter of public concern, and is in no sense a private undertaking; nor is it an effort on the part of those interested in the MONTHLY to rescue it from impending bankruptcy. The MONTHLY is in sound financial condition and is seeking no rescue measures. Its friends and supporters are interested in this new movement for the same reasons which actuate the rest of the signers to the call for the organization meeting; namely, a sincere desire to promote the course of mathematics in this country in all its many and varied aspects, and especially in that field that has been so greatly neglected,—the field of collegiate mathematics.

HISTORY OF ZENO'S ARGUMENTS ON MOTION:

PHASES IN THE DEVELOPMENT OF THE THEORY OF LIMITS.

By FLORIAN CAJORI, Colorado College.

IX.

E. POST-CANTORIAN DISSENSIONS.

Very frank and brilliant in its mode of exposition but perhaps lacking the originality and depth of the writings of Dedekind and Georg Cantor is the *Allgemeine Functionentheorie* of Paul du Bois-Reymond, Tübingen, 1882. This book, which is contemporaneous with Cantor's creation, discusses the philosophy and theory of the fundamental concepts of quantity, limit, argument, function. He declares that the difficulties surrounding the idea of a limit are not mathematical in character but have their roots in the "simplest parts of our thinking, our conceptions or images" (Vorstellungen). He says that there are two conceptions, those of the idealist and those of the empiricist, "which have equal right to count as fundamental views of rigorous science,"[1] for neither yields contradictory results, at least in pure mathematics. The author presents both sides "with equal rigor," and does not award victory to either. The idealist defends the existence not only of what can be imagined, but also of things unimaginable;[2] he assumes a transcendental attitude. Accordingly he assumes the termination of series, such as those given by endless decimal fractions, which are really "given"

[1] Paul du Bois-Reymond, *Die Allg. Functionentheorie*, Tübingen, 1882, p. 2.
[2] P. du Bois-Reymond, *op. cit.*, pp. 110, 111.

only to a certain term; he recognizes the limit of these decimals. The empiricist says on the other hand, "everything that is scientifically established springs from sense-perception; whatever is unimaginable must be rejected; we must proceed from images to images (von Vorstellungen zu Vorstellungen);[1] he denies the existence of a limit of an endless decimal fraction and is satisfied to take into account the members of the decimal as far as you may wish. A continuum such as that of G. Cantor and Dedekind does not exist, because it cannot be imagined. Take the endless decimal $0.\alpha_1\alpha_2\alpha_3\cdots$,

"In a drawing . . . imagine the ends of the distances $0.\alpha_1$, $0.\alpha_1\alpha_2$, $\cdots$ marked off from the zero point and designated by fine, short lines drawn across the line segment. These cross lines become more and more dense, and they must sometime stop, because the means of drawing fail us and, at any rate, one cannot draw an unlimited number of marks. There follows now after a short interval following the mark $0.\alpha_1\alpha_2\cdots\alpha_p$, which we shall call the fog-interval, a fine mark, somewhat longer than the previous ones, which represents the limit. In the fog-interval before the limit all sorts of philosophic apparitions play their pranks. Here play the celebrated sophisms; it is this that the idealist resolves into the *quantitates infinitesimas*. The empiricist's drawing looks a little different. The marks are continued, even if the means of drawing do not permit the marks to remain distinct. Then the marks run into each other, to show that their width is greater than the distance of $0.\alpha_1\alpha_2\cdots\alpha_p$ from the limit. The drawing of further marks is stopped when a further advance of the marks can no longer be recognized and new marks would be drawn through the same place. That is then the limit."[2]

The process of a variable continually approaching but never reaching its limit is characterized by du Bois-Reymond as follows:[3]

"Between the conception of the idealist, who lets dx be infinitely small and connects it with the idea of something at rest, unchanging, and my conception which assumes dx finite and sufficiently small but likewise at rest, there slips in a third conception (mentioned page 83, 84) in which, as is commonly expressed, dx is a quantity in the act of disappearing (quantité évanouissante), hence a quantity which is continually varying toward zero. The idea of a quantity in continual flux is repugnant to me. It goes against my grain to have symbols in my formulas for quantities which set themselves in motion as soon as I look at the formulas, and hasten toward zero, which, however, they are permitted to reach only at the end of the computation. As long as the book is closed, there reigns profound silence. No sooner do I open it than there begins that race toward zero of all quantities affected by a d."

Feelings of this sort have doubtless come to many mathematicians of the older school. This perpetual approach without ever reaching the goal is wearisome. Why not divorce the variable and its limit from the limitations of time? Or if, as in mechanical problems, the idea of time seems difficult to eliminate, hasten the successive steps of approach to the limit, the times for these steps diminishing at a sufficiently rapid rate, so that the time elements form together a converging series. Thereby the limit is reached in a finite time. The "Achilles" is a concrete illustration of such approach and reach of the limit.

Eight years after the issue of du Bois-Reymond's book there appeared a posthumous monograph prepared by a privat-docent in philosophy at the University of Strassburg, Dr. Benno Kerry. It is a philosophic study of the theory of limits.[4] Familiar with the book of du Bois-Reymond, as well as with

[1] P. du Bois-Reymond, *op. cit.*, p. 149.
[2] P. du Bois-Reymond, *op. cit.*, pp. 122, 123.
[3] P. du Bois-Reymond, *op. cit.*, pp. 140, 141.
[4] Dr. Benno Kerry, *System einer Theorie der Grenzbegriffe. Ein Beitrag zur Erkenntnisstheorie.* Herausgegeben v. Gustav Kohn, Leipzig u. Wien, 1890.

the work of Dedekind and G. Cantor, Kerry produced a very illuminating publication. Neither Kerry nor du Bois-Reymond discuss Zeno directly.

In 1885 a discussion of Zeno's arguments against motion was opened up in France, which lasted over a decade and in which a large number of writers participated. At no other decade and in no other country was the discussion of the topic so persistent and general. Never before have philosophical writers based their solutions of Zeno's puzzles so persistently upon the postulate of the discontinuity of time and space. Among the participants in the discussion were the mathematicians Paul Tannery, G. Milhaud, G. Frontera, G. Mouret, and L. Couturat, but the leading part was taken by the French philosophers Ch. Renouvier, F. Evellin, G. Noël, V. Brochard, and G. Lechalas.

In 1885 the noted historian of mathematics, Paul Tannery, published an article, *Le concept scientifique du continu. Zénon d'Elée et Georg. Cantor*,[1] which we noted at the beginning of this history as advancing a brilliant and novel explanation of the *purpose* of Zeno's arguments. It concluded with an account of Cantor's continuum. This article was supported by a paper from the pen of Gaston Samuel Milhaud, who was successively professor of mathematics in the lycées at Nizza, Havre, Lille and Montpellier and who has been, since 1895, professor of philosophy at the University of Montpellier. Otherwise Tannery's article received little attention. The discussions carried on during the decade do not center in Cantor's continuum. Cantor's ideas were then very recent and not yet fully elaborated. Milhaud's article above referred to bore the title, *Le concept du nombre chez les Pythagoriciens et les Eléates.* Before this Milhaud had written a philosophical dialogue on *La notion de limite en mathématiques*,[2] which avowedly aimed to throw light on the "Achilles." *A* and *B* discuss the subject. *A* declares that the mind cannot encompass at once an unlimited series of elements; *B* declares that this negative quality of our mind has nothing to do with the existence or non-existence of an attainable limit. Whether the limit is actually reached or not is of no interest to the mathematician and is foreign to mathematics. Though a clever dialogue, it was criticized by Ernest Jean Georges Mouret, ingénieur-en-chef de ponts et chaussées, for dodging the real issue in the "Achilles" and failure to remove the underlying difficulties.[3]

Victor Brochard wrote on Zeno in 1885, in a memoir to l'Académie des Sciences morales et politiques and again in 1893, in a paper entitled, *Les prétendus sophismes de Zénon d'Élée.*[4] He is mainly concerned with the question of Zeno's purpose and favors traditional views rather than the views of Tannery and Milhaud. In the earlier publication Brochard expressed the belief that Zeno meant actually to deny all motion. In the later article he prefers to speak of Zeno's *arguments* rather than Zeno's *sophisms.*

[1] *Revue philosophique de la France et de l'étranger.* Dixième année, XX, 1885, Paris, pp. 385–410. Excepting the part on G. Cantor, this article is reproduced almost verbatim in P. Tannery's *Pour l'histoire de la science hellène*, Paris, 1887, Chapt. X, pp. 247–261.

[2] *Revue philosophique* [Th. Ribot], Vol. 32, Paris, 1891, p. 1.

[3] "Le problème d'Achille" by George Mouret, in *Revue philosophique*, Vol. 33, Paris, 1892, p. 67.

[4] *Revue de métaphysique et de morale*, I, 1893, Paris, pp. 209–215.

A pamphlet, entitled, *Étude sur les arguments de Zénon d'Élée contre le mouvement*, Paris, 1891, was published by G. Frontera, in the preface of which he states that he was induced to prepare this study by the fact that some young students of philosophy informed him that in certain lectures on philosophy which they had attended it was seriously taught that the "Achilles" could not be explained and that motion was an illusion of the senses. Frontera holds the views of a mathematician untouched by the ideas of G. Cantor. He speaks in the "Dichotomy" of an infinite *series* of parts, instead of an infinite *number* of parts, "parce que l'idée de nombre exclut l'idée de l'infini." He says that Zeno considered only *space*, while the subject calls for the consideration of three things, *space, time, velocity*. The sum of $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., of a distance cannot be infinite as Zeno claimed, but must be finite. In dealing with series, Frontera reveals no knowledge of St. Vincent and other writers on the "Achilles," except the contemporary French writers, V. Brochard, F. Evellin. He speaks in high terms of Evellin's work, *Infini et quantité*. He writes down the convergent series for the time required to overtake the tortoise. He rightly insists that the velocities of Achilles and the tortoise must be maintained to the end, that Zeno does not do this and arrives at an absurd conclusion by the tacit assumption of chimerical conditions. The "Stade" he considers simply a question of relative motion. G. Mouret[1] criticizes Frontera for passing from the series to its limit without perceiving that it is precisely at this point where the difficulty lies. Mouret declares that the "Achilles" constitutes really a criticism of the foundations of convergent series and of the infinitesimal calculus. Frontera made reply,[2] emphasizing his former contention that the difficulty with the "Achilles" has been the chimerical hypothesis introduced by Zeno and his followers for now nearly 2,500 years.

Exception to Mouret's remark that Zeno's arguments were a criticism of the fundamental principles of the calculus and of series was taken by Louis Couturat.[3] Mouret's claim that Zeno arbitrarily excludes the point of meeting in the "Achilles" is historically not a correct interpretation; Zeno claimed that, to arrive at a point of meeting, Achilles had to run through an infinity of spaces, which Zeno thought impossible. The nerve of Zeno's argument consists, according to Couturat, in the axiom which even to-day is accepted as evident by many good heads: "The actual infinite cannot be realized"; any movement must contain an actual infinity of parts, which Zeno declared to be impossible. Zeno's sophism, if it exists, is more subtle and more profound than Mouret realizes; it is not a gross paralogism the falsity of which at once leaps into view.

In the year 1893 the discussion continued unabated. Georges Noël contributed an article, *Le mouvement et les arguments de Zénon d'Élée*,[4] which in many respects is an able discussion, but the author unfortunately does not always

[1] *Revue philosophique*, Vol. 33, 1892, p. 67.
[2] *Revue philosophique*, Vol. 33, 1892, p. 311.
[3] *Revue philosophique*, Vol. 33, 1892, p. 314.
[4] *Revue de métaphysique et de morale*, I, 1893, pp. 107–125.

distinguish between what is assumed and what is the result of clear logic, with that sharpness that is necessary to clear up such a difficult subject as motion; the article is marred by hidden assumptions. Noël claims that as soon as one grasps the true nature of movement, the refutation of Zeno is easy. Zeno divided the displacement into an infinitely increasing number of smaller displacements, standing out as distinct events which must concur in producing the final event. These smaller displacements, infinite in number, are not all given; hence Zeno concludes that the final event cannot take place. But, says Noël, the smaller displacements are not at all veritable conditions for the occurrence of the final event. They are coördinate events, but not subordinate. The final event derives its raison d'être direct from the state of the motion and its velocity. Thereby all positions taken by the moving point are given and all are on an equal footing. Their order of succession in time is in no way an order of logical dependence. Logically they are all given with the motion. Little does it matter if they are infinite in number. Neither the moving body, nor the mind contemplating it is really bound to number them. They do not introduce any real divisions in the motion. It is one motion, and its continuity excludes all actual division. The particular motions do not exist, except from an arbitrary subjective view point. Thus the "Dichotomy" and "Achilles" are real sophisms, but of such a nature that the human mind is almost inevitably carried away by them, as long as it neglects to subject the fundamental principles to a critical analysis. Motion is not a succession of positions, it is a "becoming."

Noël attacks the advocates of discontinuity, such as Evellin, who postulate the existence of a minimum distance and minimum time. Noël argues that such minimum existence is disproved by the "Arrow" and the "Stade."

Many years before this, Ch. Renouvier[1] and F. Evellin[2] argued against the actual infinite and in favor of the discontinuity of space and time; the number of parts in which the path AB can be divided is either finite or infinite, but the number cannot be infinite without the inexhaustible being found exhausted, hence the number is finite. Besides his book on *Infini et quantité*, Evellin published articles in 1893[3] and 1894[4] in which he elaborates his ideas with reference to our topic. Noël's explanation of the nature of motion does not appeal to him; Noël's hypothesis of motion as a "becoming" affords only a moment's illusion of having escaped the difficulties set by the dialectician of Elea. Evellin declares that Zeno's four arguments present two branches of a dilemma. The "Dichotomy" and "Achilles" are aimed at the mathematicians with their infinity, the "Arrow" and "Stade" are aimed at the partisans of limited division. Making negation of the infinite, he explains Zeno by assuming the discontinuity of time and space and the existence of indivisible parts of space and

[1] *Esquisse d'une classification systematique des doctrines philosophiques*, T. I, Paris, 1885. Renouvier's proof of the non-existence of the actual infinite, as well as other proofs of this are critically examined by Georg Cantor in his article, "Mitteilungen zur Lehre vom Transfiniten" in *Zeitschr. f. Philosophie und philosophische Kritik*, Bd. 91. Separatabdruck, Halle, p. 22.

[2] François Evellin, *Infini et quantité*, 1880.

[3] "Encore a propos de Zénon d'Élée," *Revue de métaphysique et de morale*, I, p. 382.

[4] "La divisibilité dans la grandeur," *Revue de métaphysique et de morale*, II, pp. 129–152.

time. On the hypothesis of unlimited divisibility he claims that Zeno successfully disproved motion. The concept of a line, says Evellin, must be such as to explain motion. The line must have parts; motion exhausts the line and therefore also its parts; hence, those parts have a number, but this number escapes us. Each moment of motion must mark an advance, but there is no advance except in the exhaustible and the finite. The "Dichotomy" and "Achilles" make this very plain. Evellin denies that the "Stade" disproves the possibility of indivisibles.

$$\begin{array}{ccc} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array} \qquad \begin{array}{ccc} a & b & c \\ a_1 & b_1 & c_1 \\ & a_2 & b_2 & c_2 \end{array}$$

If in an indivisible moment the indivisible lines represented by a, b, c, a_1, etc., shift, so that a_2 comes to c_1, the question arises, how can a_2 and b_1 have found time to meet each other? They must indeed pass each other, if the discontinuity of space is assumed, but they need not meet at all in case of discontinuity.

Another champion of discontinuity is Georges Lechalas. In 1895 he brought out a book, an *Étude sur l'espace et le temps*, Paris, of which an enlarged edition appeared in 1910. In 1893 he contributed a journal article on Zeno.[1] He proceeds on the maxim that, so far as the realized number of things in the actual world is concerned, number always means *finite* number. He distinguishes abstract from realized number. He made a study of G. Cantor and offers no objection to actual infinity, provided this concept is confined to abstract number. He claims contradiction in its realization. All realized aggregates, he says, can be counted by the ordinary process. He denies the possibility of a real continuum and hence concludes that both space and time are discontinuous. While agreeing with Evellin on matters of discontinuity, Lechalas disagrees with him on the existence of a minimum distance and a minimum time. Evellin's mode of escape from the clutches of the "Stade" does not satisfy him. If the atom is indivisible, says Lechalas, this is not a minimum of extension; extension is not a property of atoms, but a relation between them. A point in moving from one position to another occupies only a finite number of intermediate positions; hence, in the "Stade," a_2 and b_1 may not be in the same vertical line at any time. Using discontinuity as a magic wand, he gets a_2 to c_1 without meeting b_1.

[To be concluded in the November issue.]

[1] "Note sur les arguments de Zénon d'Élée" in *Revue de métaphysique et de morale*, I, 1893, p. 396.

THE THEORY OF POLES AND POLARS.

By M. B. PORTER, University of Texas.

In the text-books on analytic geometry the symmetry of the equation

$$\Sigma a_{ij}x_ix_j' = 0 \quad [P]$$

is used to show that $[P]$ is the equation of the chord (plane) of contact of the point x' with regard to the conic (quadric)

$$Q = \Sigma a_{ij}x_ix_j = 0,$$

while the theorems relating to secant lines (planes) are usually deferred until projective properties are taken up—usually in a separate course.

In addition to necessitating a preliminary treatment of tangent lines and planes, this method has the inconvenience that it does not give the properties which make ruled constructions of the tangents possible.

It is the purpose of this note to show that the various geometric properties given by the vanishing of the bilinear covariant,

$$\Sigma a_{ij}x_ix_j' = 0,$$

can be deduced in an elementary and simple manner from the equation itself, and in a way that is applicable without modification to quadratic loci in any number of dimensions.

The treatment here given is for quadrics in three dimensions and the coördinates used are point coördinates, though the processes admit, of course, the usual dual interpretation.

1. Writing our quadric in the form

$$\begin{aligned} Q = \sum_1^4 a_{ij}x_ix_j \\ \equiv (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4)x_1 + (a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4)x_2 \\ + (a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4)x_3 + (a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4)x_4 \\ \equiv A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 = 0, \end{aligned}$$

where $a_{ij} = a_{ji}$ and $A_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4$. Thus $\Sigma a_{ij}x_ix_j' = 0$ is obtained by priming either the x's inside or the x's outside the parentheses.

A *vertex* is a set of x's which satisfy the four equations

$$A_i = 0, \; i = 1, 2, 3, 4, \tag{1}$$

and the quadric is non-singular if it has no vertex (system (1) of rank 4), a *cone* if it has one vertex (system (1) of rank 3), a *plane-pair* if it has a line of vertices (system (1) of rank 2), a double plane if it has a plane of vertices (system (1) of

rank 1). From this we have that *P vanishes identically if x' is a vertex, and passes through all the vertices if x' is not a vertex.*

2. Consider now the pencil of quadrics

$$\overline{Q} = k_1 Q + k_2 Q_1 = 0, \tag{2}$$

where

where $Q_1 = \Sigma b_{ij} x_i x_j$, and the polar of $\overline{Q}$ with respect to x' is

$$k_1 \Sigma a_{ij} x_i x_j' + k_2 \Sigma b_{ij} x_i x_j' = 0, \tag{3}$$

which is the equation of a pencil of planes. If $\overline{Q} = 0$ is a singular quadric of the pencil (2) and x' is one of its vertices, [3] vanishes identically, so that

$$P \equiv k_1 \Sigma a_{ij} x_i x_j' \equiv - k_2 \Sigma b_{ij} x_i x_j' = 0.$$

Theorem: *thus we have that the polars of x' with respect to all the quadrics of pencil* (2) *are the same, if x' is a vertex of a quadric of the pencil. Moreover, P will in this case pass through the vertices of all other singular quadrics of the pencil.*

3. The application of this theorem leads at once to the geometric properties in question.

First let Q_1 be two planes, Q_1 is then a singular quadric of the pencil with a line of vertices which we shall suppose passes through the point x'. The polar of x' thus passes through the vertices of all the *other* singular quadrics of the pencil, *i. e.*, through the vertices of the two cones passing through the two conics cut out of $Q = 0$ by $Q_1 = 0$. If now one of the planes of which $Q_1 = 0$ consists becomes tangent to $Q = 0$ the vertex of one of these cones will become the point of contact of this tangent plane, so that $P = 0$ is the equation of the plane of contact of x'. If now the point x' approach the surface of $Q = 0$ as limit, $P = 0$ approaches the tangent plane so that if x' is on $Q = 0$, $P = 0$ is the tangent plane at x'.

Furthermore, if x' is any point on the line of vertices of $Q_1 = 0$,

$$P = \Sigma a_{ij} x_i (k_1 x_j' + k_2 x_j'') = 0$$

will determine a pencil whose axis,

$$\Sigma a_{ij} x_i x_j' = 0, \Sigma a_{ij} x_i x_j'' = 0,$$

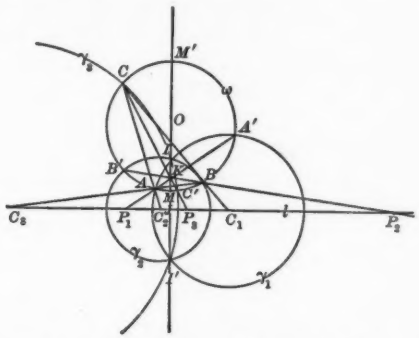
is the conjugate line of $(x'x'')$.

Secondly, if Q_1 is a double plane, its pole lies in P. More generally if $Q_1 = 0$ is any cone with vertex at x', $P = 0$ passes through the vertices of the three remaining cones which pass through the twisted quartic in which $Q = 0$ and $Q_1 = 0$ intersect. The existence of the remaining cones is shown, of course, by setting the discriminant of $\overline{Q}$ equal to zero.

ON THE CIRCLES OF APOLLONIUS.

By NATHAN ALTSHILLER, University of Colorado.

DEFINITION. The interior and exterior bisectors of the angles A, B, C of a triangle ABC meet the opposite sides in the points U, U'; V, V'; W, W' respectively. The circles described on the segments UU', VV', WW' as diameters, are called the *circles of Apollonius.*



Not all the lines considered are drawn in the figure.

(1) The bisectors CW, CW' being perpendicular to each other, the segment WW' subtends a right angle at the vertex C, therefore C is on the circle γ_3 described on WW' as diameter. Similarly for the points A and B with respect to the circles γ_1 and γ_2 described on UU' and VV' as diameters, and therefore:

The Apollonian circles pass through the respective vertices of the triangle.

(2) The bisectors CW, CW' separate harmonically the lines CA, CB,[1] hence any point S of the circle γ_3 is the center of the harmonic pencil of lines $S(ABWW')$, and since SW, SW' are perpendicular to each other, they are the bisectors of the angles at S[1]. According to a well-known theorem of plane geometry, we have for the triangles ABC and ABS,

$$\frac{CA}{CB} = \frac{AW}{BW} = \frac{AW'}{BW'}; \quad \frac{SA}{SB} = \frac{AW}{BW} = \frac{AW'}{BW'}.$$

Hence

$$\frac{SA}{SB} = \frac{CA}{CB}.$$

Now, if for any point S' in the plane we have $S'A/S'B = CA/CB$, the bisectors of the angles at S' will pass through W, W', according to the converse of the theorem cited, and the segment WW' will subtend at S' a right angle, *i. e.*, S' is on γ_3. These considerations may be repeated for the circles γ_1 and γ_2. Hence:

[1] John W. Russell, *Elementary Treatise on Pure Geometry*, pp. 18, 19.

The circle of Apollonius is the locus of a point the ratio of whose distances from two fixed points is constant.[1]

This property is frequently taken as the definition of the circle of Apollonius.

(3) From the definition and (1) it is evident that any two of the circles of Apollonius cross each other. Now, if I, I' are the points common to any two of these circles, say γ_2, γ_3, we have (2)

$$\frac{IA}{IC} = \frac{BA}{BC}; \quad \frac{IA}{IB} = \frac{CA}{CB}.$$

Hence

$$\frac{IB}{IC} = \frac{BA}{CA}, \text{ i. e., } I \text{ is on } \gamma_1.$$

The same being true for the point I', we infer that

The three circles of Apollonius have two points in common.[2]

COROLLARY. *The centers of the three circles of Apollonius are collinear.*

(4) The points A, B, W, W' being harmonic (2) and WW' being a diameter of γ_3 (1), any circle passing through A, B cuts the circle γ_3 orthogonally.[3] Similarly for γ_1 and γ_2. Hence:

The circumcircle is orthogonal to each of the three circles of Apollonius.

(5) The radii OA, OB, OC of the circumcircle ω are the tangents drawn from the center O of ω to the respective circles of Apollonius (1, 4), and since $OA = OB = OC$, the point O belongs to the radical axis of the circles $\gamma_1, \gamma_2, \gamma_3$, which is the line joining the points I, I' (3) common to the three circles. Consequently:

The common chord of the three circles of Apollonius passes through the center of the circumcircle.[2]

COROLLARY.[3] *The points of intersection M, M' of the circumcircle with the chord $s \equiv II'$ are harmonically separated by the points I, I'.*

(6) The circles ω and γ_3 having the point C in common (1) and being orthogonal (4), the tangent to ω at C passes through the center C_3 of γ_3. The centers C_1, C_2 of the circles γ_1, γ_2 are determined in a like manner. Hence:

The tangents to the circumcircle at the vertices of the triangle meet the opposite sides of the triangle in the centers of the respective circles of Apollonius.[2]

(7) The circles ω and γ_3 intersect in C (1); let C' denote their other point of intersection. Since OC, OC' are the tangents drawn from O to γ_3 (4), the chord $s_3 \equiv CC'$ is the polar of O with respect to γ_3, and therefore meets $s \equiv$ chord II' at the harmonic conjugate K of O with respect to the pair of points I, I'. Similarly for the chords of intersection $s_1 \equiv AA'$, $s_2 \equiv BB'$ of ω with the circles γ_1 and γ_2. Therefore:

The three chords joining the three pairs of points of intersection of the circum-

[1] Weber and Wellstein, *Encyklopädie der Elementar-Matematik*, Vol. 2, p. 250, sec. ed.
[2] John Casey, *Analytic Geometry*, p. 146, sec. ed. This MONTHLY, February, 1915, p. 59.
[3] Russell, *loc. cit.*, p. 26.

circle with each of the three circles of Apollonius, meet on the radical axis of the Apollonian circles.[1]

The common point K is called the *Lemoine* or *Symmedian point* of the triangle; the chords s_1, s_2, s_3 are called the *Symmedian lines* or *Symmedians* of the triangle;[2] $s \equiv II'$ is called the *Brocard diameter.*[3]

(8) The tangents from C to ω are the lines C_3C and C_3C' (4), C_3 is therefore the pole of $s_3 \equiv CC'$ with respect to ω; since C_3 is on AB, the line s_3 passes through the pole of AB with respect to ω, which pole is the point of intersection of the tangents to ω at A and B. Similarly for the chords s_1, s_2. Hence:

The Symmedian lines pass through the respective vertices of the triangle formed by the tangents to the circumcircle at the vertices of the given triangle.[4]

(9) The four lines s_1, s_2, s_3, s meeting at K (7) have their respective poles C_1, C_2, C_3, ∞ (8) with regard to the circumcircle ω on the polar of K with respect to ω, and the anharmonic ratio of the four lines is equal to the anharmonic ratio of the four points.[5] Hence:

A. The line of centers of the Apollonian circles is the polar of the Lemoine point with respect to the circumcircle.

The line is called the *Lemoine axis* or *line.*[6]

B. The anharmonic ratio of the three Symmedians and the Brocard diameter is numerically equal to C_1C_3/C_2C_3.

(10) The Lemoine axis 1 is perpendicular to s at the mid-point J of the segment II' (3), and since the segments MM' and II' are harmonic (5, cor.), the point J lies without the segment MM'[7]. Consequently:

The points of intersection of the Lemoine axis with the circumcircle are always imaginary.

COROLLARY. *The Lemoine point of a triangle lies always within the circumcircle* (9*A*).

(11) Since the line s passes through the pole O, with regard to γ_3, of the chord CC' (7), the pole P_3 of s with respect to γ_3 lies on CC'. On the other hand P_3 lies on 1, since s is perpendicular to the diameter 1 of γ_3. Similarly for the poles P_1, P_2 of s with respect to γ_1, γ_2. Hence:

The Symmedians meet the Lemoine axis in the respective poles of the Brocard diameter with regard to the Apollonian circles.

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[1] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[2] Casey, *loc. cit.*, pp. 63, 64; Gallatly, *loc. cit.*, pp. 1, 2; R. Lachlan, *An. Elem. Treatise on Pure Geometry*, pp. 62, 63.

[3] John Casey, *Analytical Geometry*, p. 146, sec. ed. This MONTHLY, Feb., 1915, p. 59.

[4] Gallatly, *loc. cit.*, p. 5.

[5] Russell, *loc. cit.*, pp. 165, 117.

[6] William Gallatly, *The Modern Geometry of the Triangle*, p. 6.

[7] Russell, *loc. cit.*, p. 15.

A TRIBUTE TO MILDRED LENORA SANDERSON.

The remarkable mathematical ability and originality shown by Miss Sanderson in her master's and doctor's theses and the very unusual ease with which she assimilated ideas in all branches of pure and applied mathematics, combined with her enthusiasm for that science, gave full promise of a highly successful career for her in research. Her death on October 15, 1914, only a year after completing her graduate studies, was not only a distinct loss to progress in mathematical research in America, but was a very keen blow to her fellow students, to all of whom she had endeared herself by her most lovable personality.

Miss Sanderson was born May 12, 1889, in Waltham, Mass., on the place where her ancestors had lived for over 200 years. She graduated from the North Grammar School at 13 and was valedictorian of her class in the Waltham High School. One of her instructors in the latter school writes that "Miss Sanderson was gentle-mannered, of brilliant intellect, an exact student, broad-minded, self-reliant, and courageous." She entered Mt. Holyoke in 1906, received "Sophomore Honors" in June, 1908, for general scholarship, and "Senior Honors" in mathematics at graduation in 1910. She held the Bardwell Memorial Fellowship for 1910 and 1911, and began her graduate work in mathematics at the University of Chicago, taking the degree A. M. in 1911. The subject of her master's thesis was "Generalizations in the Theory of Numbers and Theory of Linear Groups." Of this original and valuable thesis a very brief extract was printed in the *Annals of Mathematics*, Ser. 2, Vol. 13, 1911, pp. 36–39. This work might well have served for her doctor's thesis; but she was quite willing to undertake a new investigation in a wholly different field. The resulting thesis for the degree Ph.D., taken at the University of Chicago in 1913, was entitled "Formal Modular Invariants with Application to Binary Modular Covariants," and appeared in the *Transactions of the American Mathematical Society*, Vol. 14, 1913, pp. 489–500. This paper is a highly important contribution to this new field of work; its importance lies partly in the fact that it establishes a correspondence between modular and formal invariants. Her main theorem has already been frequently quoted on account of its fundamental character. Her proof is a remarkable piece of mathematics.

During the year 1911–13, she held a fellowship in mathematics in the University of Chicago. From October, 1913, to February, 1914, when her final illness began, she was instructor in mathematics in the University of Wisconsin.

If I may be permitted to add my personal tribute to the universally expressed tribute to her remarkable ability, it would be to say that she was my most gifted pupil.

L. E. DICKSON.

THE UNIVERSITY OF CHICAGO,
June, 1915.

BOOK REVIEWS.

EDITED BY W. H. BUSSEY, University of Minnesota.

Plane Geometry. By JOHN H. WILLIAMS and KENNETH P. WILLIAMS. Lyons and Carnahan, Chicago, 1915. 264 pages.

At the outset an effort is made to justify the study of geometry. This is usually a vital question in the minds of young people when they undertake the subject for the first time.

Scattered through the book will be found exercises in construction, numerical problems, and original exercises. These appear as soon as theorems or definitions have been given that lead up to them. Numerical exercises and original theorems, if brought in as soon as possible, will aid materially in fixing fundamental notions without, at the same time, taking up the usual theorems too rapidly. In this respect the book is well conceived. Following the definitions of complementary and supplementary angles on page 21, is found a well-chosen list of numerical problems, also following the theorem on the "sum of the angles of a triangle" will be found appropriate numerical exercises.

At the close of each book is a review which summarizes, in concise form, the main notions of the book. This includes definitions, axioms, postulates, corollaries, theorems, etc., in question form. For example: When is one proposition the converse of another? How are triangles classified as to their sides? How are the acute angles of a right triangle related? These and many others constitute a complete summary of the first book. This book closes with the significant statement: "Name three general reasons for studying geometry."

It would seem that the field for plane geometry had been so thoroughly covered that there could be no excuse for new texts; however modern conditions and modern thought call for some changes. The school curriculum is more crowded than formerly and pupils take up the study of geometry at a somewhat earlier age; and the tendency toward the practical makes it necessary for a geometry text to justify itself from the standpoint of practicability, besides being somewhat simplified to meet these changed conditions. To hold its own with the pressing demands of other subjects, geometry must be made interesting. Variables and limits are not introduced in Book II, as is usual, but instead in Book V just before the theorems on circumference and area of circles. This arrangement seems very satisfactory; the theory of limits is needed only in connection with circles. Earlier theorems on incommensurables may with propriety be omitted.

On page 88 is found the following corollary: "The ratio of two incommensurable arcs of the same circle or of equal circles can be expressed as the ratio of two commensurable arcs to any degree of exactness, if a sufficiently small unit of measure is used;" with a note to the effect that in practice there is no distinction to be made between commensurable and incommensurable lines as the fraction to be dropped from one of two incommensurable lines to make the lines commensurable is less than the error made in the work of measuring.

On the whole this text seems well adapted to present needs and should meet with a generous response.

North High School,
Minneapolis.

F. W. Gates.

New Plane Geometry. By Edward Rutledge Robbins. American Book Company, New York, 1915. 264 pages.

This book is a revision of a Plane Geometry published by the same author a few years ago. The revision, as the author suggests, is an outgrowth of the author's experience and suggestions from teachers who have used the former edition as well as from recommendations of the "National Committee of Fifteen."

While the general plan of the new book is much the same as the old, there are many additions that make the book comply with recent demands.

Each page is attractive, the material is well arranged, statements of theorems and the words *given, to prove, proof,* etc., are in italics.

Simple fundamental truths are explained instead of being formally demonstrated," as for example, "all straight angles are equal." Several such will be found on pages 13 and 14.

No theorems are demonstrated in full. Proofs are given in outline, reasons indicated by references. It would seem that if a few theorems were demonstrated in full the pupil might have a more definite notion of what constitutes a complete demonstration.

Original exercises are scattered through the book in abundance. These appear as early as possible. After theorem (2) will be found eleven original exercises; after theorem (34), 28 exercises; after theorem (35), 11 more; and so on.

These exercises do not depend upon previous exercises for proofs, the numbered references alone being sufficient.

The large number of original exercises makes it possible to furnish the ambitious pupil with an abundance of choice material.

In all there are about 200 original exercises in Book I, especially of the theorem variety. Very few numerical problems are to be found in this book; possibly a number of such problems would have been an advantage.

Book II is unusually attractive. Definitions are well illustrated. A large number of original exercises are found in this book. These must be seen to be appreciated.

At the close of Book III will be found a collection of 69 exercises (numerical). These involve the application of a large number of theorems of every description. These exercises are followed by 53 original theorems, and these by 43 original constructions.

At intervals are historical notes which help to interest the pupil in the subject.

Summaries found on pages 68, 94, and 180 classify the theorems in a way that will be helpful for reference purposes.

This text seems to be an improvement over the author's earlier edition, both in attractiveness of the printed page and abundance of well-selected original exercises. After all the pupil's ability to solve original theorems is the real test of geometrical knowledge.

The text as a whole seems to meet very well the recent demands for the subject.

North High School,
Minneapolis.

F. W. Gates.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel and R. P. Baker.

PROBLEMS FOR SOLUTION.

ALGEBRA.

441. Proposed by W. D. CAIRNS, Oberlin College.

Prove that the equation $(e-1)x = e^x - 1$ has two and only two real roots. [Adapted from *L'Intermédiaire*.]

442. Proposed by CLIFFORD N. MILLS, Brookings, North Dakota.

Show that the sum of n terms of the series $1/2 - 1/3 + 1/4 - 1/6 + 1/8 - 1/12 + \cdots$ is $1/3[1 - (1/2)^{n/2}]$ when n is even, and $1/3[1 + 2\sqrt{2}(1/2)^{(n/2)+1}]$ when n is odd.

GEOMETRY.

472. Proposed by PAUL CAPRON, U. S. Naval Academy.

The sides of a spherical triangle are a, b, c; the corresponding opposite angles are A, B, C; p and P are the polar distances of the inscribed and circumscribed circles; $a+b+c = 2s$; $A+B+C = 2S$. From a geometric figure, by the formula for solving right spherical triangles, show that

$$\tan^2 p = \operatorname{cosec} s \sin(s-a) \sin(s-b) \sin(s-c); \tag{1}$$

$$\cot^2 P = -\sec S \cos(S-A) \cos(S-B) \cos(S-C). \tag{2}$$

Thus establish the usual formulas for the tangent of the half-sides and half-angles.

Also show that

$$\frac{\text{sine of angle}}{\text{sine of the opposite side}} = \frac{\cot P \cos S}{\tan p \sin s}. \tag{3}$$

473. Proposed by FRANK R. MORRIS, Gendale, Calif.

What is the length of the longest rectangle an inch wide that can be placed inside another rectangle 12 inches long and 8 inches wide. Obtain the result correct to the third decimal.

CALCULUS.

393. Proposed by LAENAS G. WELD, Pullman, Ill.

Find the area of the least ellipse which can be drawn upon the face of a brick wall so as to inclose four bricks.

394. Proposed by W. W. BURTON, Macon, Ga.

A horse runs 10 miles per hour on a circular race-track in the center of which is an arc-light. How fast will his shadow move along a straight board fence (tangent to the track at the starting point) when he has completed one eighth of the circuit?

MECHANICS

[Numbers 307 and 308 were omitted in the May issue.]

307. Proposed by LAENAS G. WELD, Pullman, Illinois.

Four forces W, X, Y, and Z, concurrent in O, are in equilibrium. Prove that

$$W : X : Y : Z :: \Delta_1 : \Delta_2 : \Delta_3 : \Delta_4,$$

where

$$\Delta_1 = \begin{vmatrix} 1 & \cos XOY & \cos XOZ \\ \cos XOY & 1 & \cos YOZ \\ \cos XOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_2 = \begin{vmatrix} 1 & \cos WOY & \cos WOZ \\ \cos WOY & 1 & \cos YOZ \\ \cos WOZ & \cos YOZ & 1 \end{vmatrix}^{\frac{1}{2}};$$

$$\Delta_3 = \begin{vmatrix} 1 & \cos WOX & \cos WOZ \\ \cos WOX & 1 & \cos XOZ \\ \cos WOZ & \cos XOZ & 1 \end{vmatrix}^{\frac{1}{2}}; \quad \Delta_4 = \begin{vmatrix} 1 & \cos WOX & \cos WOY \\ \cos WOX & 1 & \cos XOY \\ \cos WOY & \cos XOY & 1 \end{vmatrix}^{\frac{1}{2}}.$$

308. Proposed by H. S. UHLER, Yale University.

Prove that when a ray of light passes obliquely through a prism in such a manner as to maintain a constant value for the total deviation of the projection of the ray on a principal section, the ray inside the prism generates a cone of elliptical right section. It is assumed that the prism is surrounded by a medium having a smaller index of refraction than the index of the material of the prism.

NUMBER THEORY.

234. Proposed by FRANK IRWIN, University of California.

Start with any number, for instance 89, and add to it the number obtained by reversing the order of its digits: $89 + 98 = 187$. Now perform the same operation on the result: $187 + 781 = 968$. If we continue in this way we arrive, after a certain number of operations, at a number which reads the same forwards and backwards (24 operations bring us to 8813200023188). Will this be the case no matter with what number we start?

Note.—I am told that this is an old problem, but do not know whether it has ever been solved. (No other number under 100, except, of course, 98, requires so many operations to lead to the desired result, as 89 does.)

SOLUTIONS OF PROBLEMS.

ALGEBRA.

409. Proposed by C. E. GITHINS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelepiped so that its diagonal shall be rational.

(*A*) REMARKS BY ARTEMAS MARTIN, Washington, D. C.

I. On page 162 of the May MONTHLY, Mr. Eells says: "I fail to see, however, how I solved a *different* problem from the one porposed."

The problem proposed is stated above; the problem solved by Mr. Eells follows:

Find integral values for the edges of a rectangular parallelepiped so that its diagonal, and the diagonal of one of its faces, shall be rational.

The problem proposed does not require the diagonal of one of the faces of the solid to be rational, and the added condition restricts its generality and limits the number of possible solutions.

II. Mr. Eells says on page 269 of the October issue, without any qualifying condition: "This gives the smallest rational parallelepiped," edges 3, 4, 12 and diagonal 13.

III. The fault in my solution was in assuming $x = a$, $y = b$, $d = z + c$. I should have put $x = na$, $y = nb$ and $d = z + nc$; then

$$(na)^2 + (nb)^2 + z^2 = (z + nc)^2 = z^2 + 2ncz + n^2c^2,$$

which gives, after dividing by n,

$$z = \frac{na^2 + nb^2 - nc^2}{2c} \quad \text{and} \quad d = \frac{na^2 + nb^2 + nc^2}{2c}.$$

Now take $n = 2c$ and we have the integral values

$$x = 2ac,\ y = 2bc,\ z = a^2 + b^2 - c^2,\ d = a^2 + b^2 + c^2;$$

therefore

$$(a^2 + b^2 + c^2)^2 = (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2 \qquad (A),$$

whatever be the values of a, b, c; and it is seen, without any elaborate proof, that x and y are always even numbers.

Permuting the letters a, b, c in (A) we get

$$\begin{aligned}(a^2 + b^2 + c^2)^2 &= (a^2 + b^2 - c^2)^2 + (2ac)^2 + (2bc)^2\\ &= (a^2 + c^2 - b^2)^2 + (2bc)^2 + (2ab)^2\\ &= (b^2 + c^2 - a^2)^2 + (2ab)^2 + (2ac)^2.\end{aligned}$$

The additional formulas, however, will only give sets of values for x, y, z that can be obtained by interchanging the numerical values assigned to a, b, c in (A).

In order to obtain values for x, y, z that have no common divisor it is necessary to impose certain restriction on a, b, c. Such values of x, y, z may be called a prime set.

1. The values given to a, b, c must not *contain* a common factor. Hence they can not all be even as in that case x, y, z would be divisible by 2.
2. If a, b, c be all odd, the numbers x, y, z will not have a common divisor.
3. If any two are even and the other odd, the numbers will not have a common divisor.
4. If any two are odd and the other even, then 2 will be even and the numbers will have the common divisor 2 since x and y are always even.
5. But the numbers will be a prime set in all cases after the common factor is divided out.

IV. Mr. Carleton does not put down the rational value of c in his solution on page 164, but it is obvious that

$$c = \sqrt{(2ab)} = \sqrt{(2a \times \tfrac{1}{2}a^2)} = a^2,$$

and that we have

$$(a)^2 + (a^2)^2 + (\tfrac{1}{2}a^2)^2 = a^2 + a^4 + \tfrac{1}{4}a^6 = a(1 + \tfrac{1}{2}a^2)^2,$$

and the numbers (a, b, c) have a common factor a; therefore

$$1^2 + a^2 + \tfrac{1}{4}a^4 = (1 + \tfrac{1}{2}a^2)^2,$$

whatever be the value of a; but, for integers, a must be even.

Fractional values could be avoided by putting $a = 2m^2$, $b = n^2$; for we would then have

$$c = \sqrt{(2ab)} = \sqrt{(4m^2n^2)} = 2mn,$$

and

$$(2m^2)^2 + (2mn)^2 + (n^2)^2 = (2m^2 + n^2)^2,$$

where m and n must be prime to each other.

See the *Mathematical Magazine*, Vol. II., No. 5 (Washington, D. C., Oct., 1891), page 72, where the writer published this method of solution. For other solutions, see the same No., pp. 71–75.

V. It has been claimed that the equation (A) above includes all possible solutions of

$$x^2 + y^2 + z^2 = 0,$$

but the writer is not at present prepared to affirm or deny that claim.

(B) REMARKS BY J. W. YOUNG, Dartmouth College.

After reading the remarks by W. C. Eells, on Algebra Problem 409, in the April number of the MONTHLY, it occurred to me that the geometric formulation for the right-triangle problem given by Klein in his *Elementarmath. v. höheren Standp. aus*, Vol. I, should yield a general solution also for the rectangular parallelepiped case. The problem is to solve in integers the equation

$$a^2 + b^2 + c^2 = d^2 \quad (> 0). \tag{1}$$

Placing $x = \frac{a}{d}$, $y = \frac{b}{d}$, $z = \frac{c}{d}$ the equation becomes

$$x^2 + y^2 + z^2 = 1. \tag{2}$$

Equation (1) will be completely solved, if we find all *rational* points on the sphere (2). $(-1, 0, 0)$ is one such point. Any straight line through $(-1, 0, 0)$ and any other rational point will have the equations

$$\frac{x+1}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}, \tag{3}$$

with λ, μ, ν integral and without a common factor; and conversely, every such line (3) will cut out a rational point, P, besides $(-1, 0, 0)$. Solving (2) and (3) we find

$$P = \left(\frac{\lambda^2 - \mu^2 - \nu^2}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\mu\lambda}{\lambda^2 + \mu^2 + \nu^2}, \frac{2\nu\lambda}{\lambda^2 + \mu^2 + \nu^2}\right).$$

If D represents the H. C. F. of $\lambda^2 - \mu^2 - \nu^2$, $2\mu\lambda$, $2\nu\lambda$, and $\lambda^2 + \mu^2 + \nu^2$, then the *general solution* of (1) is given by

$$(a, b, c, d) = \left(\frac{k}{D}(\lambda^2 - \mu^2 - \nu^2), \frac{k}{D}\cdot 2\mu\lambda, \frac{k}{D}\cdot 2\nu\lambda, \frac{k}{D}(\lambda^2 + \mu^2 + \nu^2)\right), \tag{4}$$

where k is any integer.

Y. A. Le Besgue's solutions, to which reference has already been made by Mr. Eells, yield, for $\delta = 0$, the solutions above for $K = D$. If λ, μ, ν are, *e. g.*, relatively prime, no solutions are duplicated by (4). Moreover it is then easy to see that D is equal to the H. C. F. of λ and $\mu^2 + \nu^2$ or twice this H. C. F. The above method will clearly hold for the general equation $\sum_{i=1}^{n} x_1^2 = n^2$.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r} + \frac{b}{r^2} + \frac{a+b}{r^3} + \frac{a+2b}{r^4} + \frac{2a+3b}{r^5} + \cdots,$$

in which a and b are any numbers, and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the MONTHLY.

II. SOLUTION BY THE PROPOSER.

Let us designate the above numerators as follows:

$$U_1 = a;\ U_2 = b;\ U_3 = a + b;\ U_n = U_{n-1} + U_{n-2}.$$

Also let 1, 1, 2, 3, 5, 8 $\cdots$ $u_n = u_{n-1} + u_{n-2}$ be a special sequence of the above type. Then by inspection, easily verified by induction, we have

$$U_n = au_{n-2} + bu_{n-1},\ n > 2. \tag{1}$$

And furthermore, the equation $\rho^n = \rho^{n-1} + \rho^{n-2}$ gives the two solutions

$$\rho_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \rho_2 = \frac{1-\sqrt{5}}{2}.$$

Then a sequence of powers of either ρ_1 or ρ_2 will be a sequence obeying the fundamental relation (1).

Therefore

$$\rho_1{}^n = \rho_1 u_{n-2} + \rho_1{}^2 u_{n-1},$$

$$\rho_2{}^n = \rho_2 u_{n-2} + \rho_2{}^2 u_{n-1}.$$

Since

$$\rho_1 + \rho_2 = 1;\ \rho_1 - \rho_2 = \sqrt{5};\ \rho_1{}^2 - \rho_2{}^2 = \sqrt{5}$$

we have

$$\frac{\rho_1{}^n - \rho_2{}^n}{\sqrt{5}} = u_{n-2} + u_{n-1} = u_n. \tag{2}$$

We may now combine (1) and (2) and obtain

$$U_n = a\frac{\rho_1{}^{n-2} - \rho_2{}^{n-2}}{\sqrt{5}} + b\frac{\rho_1{}^{n-1} - \rho_2{}^{n-1}}{\sqrt{5}}. \tag{3}$$

In order to find the region of convergence, we may use the ratio test:

$$\frac{U_{n+1}}{r^{n+1}} \div \frac{U_n}{r^n} \text{ reduces to } \frac{1}{r}\left[\frac{a + \dfrac{u_n}{u_{n-1}}b}{\dfrac{u_{n-2}}{u_{n-1}}a + b}\right].$$

From the fundamental relation $u_n = u_{n-1} + u_{n-2}$ we have

$$\frac{u_n}{u_{n-1}} = 1 + \frac{u_{n-2}}{u_{n-1}} \text{ and therefore } \operatorname*{Lim}_{n\to\infty}\frac{u_n}{u_{n-1}} = 1 + \operatorname*{Lim}_{n\to\infty}\frac{u_{n-2}}{u_{n-1}},$$

provided that these limits exist. But $\dfrac{u_n}{u_{n-1}}$ is the $(n-1)$st convergent of the simple nonterminating continued fraction

$$1 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \cdots,$$

and therefore it has a limit; call it K. But $\operatorname*{Lim}_{n\to\infty} u_{n-2}/u_{n-1} = 1/\operatorname*{Lim}_{n=\infty} u_n/u_{n-1}$, and so $k = 1 + (1/k)$; giving $k = (1 \pm \sqrt{5})/2$; but since all u's are positive, $k = (1+\sqrt{5})/2 = \rho$. Our test ratio now reduces, when the limit is taken, to ρ_1/r.

Hence, the series

$$\sum_{n=1}^{\infty}\frac{U_n}{r^n}$$

will converge if $|r| > \rho_1$ and will diverge if $|r| < \rho_1$; where $\rho_1 = (1+\sqrt{5})/2$. By actual substitution of ρ_1 for r in the series, it may be shown that the series will also diverge for $|r| = \rho_1$.

$\therefore$ The region of convergence is $|r| > [(1+\sqrt{5})/2)]$. Now, by making use of (3) we may obtain

$$\sum_{n=1}^{\infty} \frac{U_n}{r^n} = \frac{1}{\sqrt{5}}\left(\frac{a+b\rho_1}{\rho_1(r-\rho_1)} - \frac{1}{\sqrt{5}}\frac{a+b\rho_2}{\rho_2(r-\rho_2)}\right).$$

The right-hand member now reduces to

$$\frac{b+ar-a}{r^2-r-1}$$

when ρ_1 is replaced by $(1+\sqrt{5})/2$ and ρ_2 by $(1-\sqrt{5})/2$.

Note.—We are publishing this solution for the reason that the previously published solution referred to did not consider the question of convergency correctly, and the proper investigation of this question was the Proposer's chief reason for proposing the problem. EDITORS.

ALGEBRA.

429. Proposed by C. N. SCHMALL, New York City.

It is given that d_1, d_2, d_3 are the greatest common divisors of y and z, z and x, x and y, respectively; also that m_1, m_2, m_3 are the least common multiples of the same pairs of members. If d and m are the greatest common divisor and least common multiple, respectively, of x, y, and z, show that

$$\frac{m}{d} = \left(\frac{m_1m_2m_3}{d_1d_2d_3}\right)^{\frac{1}{2}}.$$

SOLUTION BY FRANK IRWIN, University of California.

It is evident that we can get the least common multiple of two numbers by dividing their product by their greatest common divisor:

$$m_1 = \frac{yz}{d_1},\ m_2 = \frac{xz}{d_2},\ m_3 = \frac{xy}{d_3}.$$

Similarly with the three numbers x, y, z, if we divide their product by $d_1d_2d_3$, we should have their least common multiple, except that we have divided out d once too often:

$$m = \frac{xyz}{d_1d_2d_3}\cdot d.$$

We have then:

$$\left(\frac{m_1}{d_1}\cdot\frac{m_2}{d_2}\cdot\frac{m_3}{d_3}\right)^{\frac{1}{2}} = \left(\frac{yz}{d_1^2}\cdot\frac{zx}{d_2^2}\cdot\frac{xy}{d_3^2}\right)^{\frac{1}{2}} = \frac{xyz}{d_1d_2d_3} = \frac{m}{d}.$$

Also solved by A. H. HOLMES, ELMER SCHUYLER, G. W. HARTWELL, FRANK R. MORRIS, N. P. PANDYA, HERBERT N. CARLETON, PAUL CAPRON, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

443. Proposed by C. N. SCHMALL, New York City.

A quadrilateral of any shape whatever is divided by a transversal into two quadrilaterals. The diagonals of the original figure and those of the two resulting (smaller) figures are then drawn. Show that their three points of intersection are collinear.

III. SOLUTION BY LAENAS G. WELD, Pullman, Ills.

To the triangle ABC draw the transversals MN, intersecting AB in M and AC in N, and QR intersecting AB in Q and AC in R. Then $BCRQ$, $BCNM$ and

$QRNM$ are the three quadrilaterals in question. Designate the intersection of the diagonals BN and CM by F, that of BR and CQ by G, and that of QN and RM by H. Then F, G, and H are collinear?

In vectors, let $AB = \beta$, $AC = \gamma$; $AM = m\beta$, $AN = n\gamma$; $AQ = q\beta$, $AR = r\gamma$. Then

$$MC = -m\beta + \gamma \quad \text{and} \quad MF = x(-m\beta + \gamma),$$

in which x is to be determined. Also

$$NB = -n\gamma + \beta \quad \text{and} \quad NF = y(-n\gamma + \beta),$$

in which y is to be determined. Now

$$m\beta + x(-\beta m + \gamma) = AF = n\gamma + y(-n\gamma + \beta); \tag{1}$$

or

$$m(1-x)\beta + x\gamma = n(1-y)\gamma + y\beta,$$

in which the coefficients of β and γ may be equated. Thus

$$mx + y = m,$$

$$x + ny = n.$$

Solving:

$$x = \frac{n(m-1)}{mn-1},\ y = \frac{m(n-1)}{mn-1};$$

whence, by Eq. (1),

$$AF = \frac{m(n-1)}{mn-1}\cdot\beta + \frac{n(m-1)}{mn-1}\cdot\gamma. \tag{2}$$

Similarly:

$$AG = \frac{q(r-1)}{qr-1}\cdot\beta + \frac{r(q-1)}{qr-1}\cdot\gamma, \tag{3}$$

and

$$AH = \frac{mq(n-r)}{mn-qr}\cdot\beta + \frac{nr(m-q)}{mn-qr}\cdot\gamma. \tag{4}$$

By the theory of vectors, if there are three multipliers, λ, μ, ν, such that

$$\lambda\cdot AF + \mu\cdot AG + \nu\cdot AH = 0,$$

while

$$\lambda + \mu + \nu = 0,$$

the vectors AF, AG, AH terminate collinearly. This requires, by Equations (2), (3), (4), the consistency of the equations

$$\frac{m(n-1)}{mn-1}\cdot\lambda\beta + \frac{q(r-1)}{qr-1}\cdot\mu\beta + \frac{mq(n-r)}{mn-qr}\cdot\nu\beta = 0,$$

$$\frac{n(m-1)}{mn-1}\cdot\lambda\gamma + \frac{r(q-1)}{qr-1}\cdot\mu\gamma + \frac{nr(m-q)}{mn-qr}\cdot\nu\gamma = 0,$$

$$\lambda + \mu + \nu = 0.$$

This consistency is established by the vanishing of the determinant

$$\Delta \equiv \begin{vmatrix} m(n-1) & q(r-1) & mq(n-r) \\ n(m-1) & r(q-1) & nr(m-q) \\ mn-1 & qr-1 & mn-qr \end{vmatrix};$$

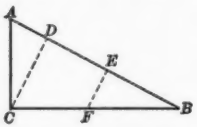
which proves the proposition.

Note.—Solutions of this problem appeared in the January and May numbers of the MONTHLY, presenting two different methods of attack. The one here given presents still a different method.

EDITORS.

459. Proposed by C. N. SCHMALL, New York City.

In a right triangle ABC, right-angled at C, a point F is taken in the side CB, and perpendiculars CD and FE are dropped on the hypothenuse AB.



Prove that

$$AD \cdot AE + CD \cdot EF = \overline{AC}^2.$$

SOLUTION BY HERBERT N. CARLETON, West Newbury, Mass.

Triangles ACD, CDB, EFB, and ACB are similar. Hence,

$$AD : CD = EF : EB;$$

whence

$$AD \cdot EB = CD \cdot EF.$$

Similarly

$$AD : AC = AC : AB;$$

whence

$$\overline{AC}^2 = AB \cdot AD = (AE + EB)AD = AD \cdot AE + AD \cdot EB.$$

Substituting for $AD \cdot EB$, we have

$$\overline{AC}^2 = AD \cdot AE + CD \cdot EF.$$

Also solved by A. M. HARDING, J. W. CLAWSON, ELMER SCHUYLER, FRANK IRWIN, WALTER C. EELLS, L. G. WELD, J. A. CAPARO, A. H. HOLMES, NATHAN ALTSHILLER, G. W. HARTWELL, HORACE OLSON, and HARMON ANNING.

460. Proposed by J. W. CLAWSON, Ursinus College, Pa.

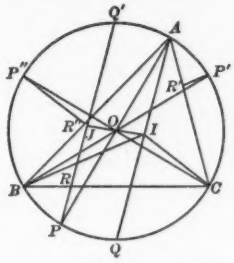
ABC is a triangle, O and I the centers of the circum- and in-circles respectively, and I', I'', I''' the centers of the three escribed circles. If AO, BO, CO meet the circumcircle in P, P', P'' respectively, and PR, $P'R'$, $P''R''$ are drawn parallel respectively to AI, BI, CI to meet BC, CA, AB respectively in R, R', R'', prove that: (1) PR, $P'R'$, $P''R''$ are concurrent, say at J. (2) $JO = OI$. (3) JI', JI'', JI''' are perpendicular respectively to BC, CA, AB. (4) AR, BR', CR'' are concurrent.

I. SOLUTION BY THE PROPOSER.

(1) Extend IO its own length to J. Join JP. Then $\triangle$'s AOI, POJ are congruent. Hence, $\angle APJ = \angle PAI$. Hence J lies on PR. Similarly, J can be shown to lie on $P'R'$ and on $P''R''$.

(2) By construction, $JO = OI$.

(3) Extend AI to meet the circumcircle at Q. Then Q is the center of a circle passing through B, C, I, I'. Hence, Q bisects II'. Also O bisects IJ. Hence, $I'J$ is parallel to OQ. But OQ is perpendicular to BC, since arc BQ = arc QC. Therefore, JI' is perpendicular to BC.



Similarly, JI'' and JI''' are perpendicular to CA and AB, respectively.

(4) Extend PJ to meet the circumcircle at Q'. It is easily shown that Q' is the opposite extremity of the diameter through Q. Then $\angle BPQ' = \angle Q'PC$.

$$\therefore \quad \frac{BR}{RC} = \sqrt{\frac{4r^2 - c^2}{4r^2 - b^2}}, \qquad \frac{CR'}{R'A} = \sqrt{\frac{4r^2 - a^2}{4r^2 - c^2}}, \qquad \frac{AR''}{R''B} = \sqrt{\frac{4r^2 - b^2}{4r^2 - a^2}},$$

where r = radius of circumcircle, and a, b, c are the sides of the triangle.

Multiplying,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1.$$

Hence, by the converse of Ceva's Theorem, AR, BR', CR'' are concurrent.

II. Solution by N. P. Pandya, Sojitra, India.

(1) The quadrilateral $ABPP'$, having as diagonals diameters of the circumcircle, is a rectangle. Hence $AB = PP'$, $AB \parallel PP'$; and similarly,

$$BC = P'P'', \; BC \parallel P'P'', \; CA = P'P'' \text{ and } CA \parallel PP''.$$

$\therefore$ $\triangle PP'P''$ and ABC are congruent and have their sides respectively parallel.

$\therefore$ $\angle P'PP'' = \angle ABC$ and PR bisects $\angle P'PP''$.

Similarly, $P'R'$ and $P''R''$ are bisectors of the remaining angles of $\triangle PP'P''$. Hence, PR, $P'R'$, $P''R''$ are concurrent at J.

(2) As AB, PP'; BC, $P'P''$; CA, $P''P$ are symmetrically situated with respect to O, then I and J are also so situated; hence, $IO = JO$, J being the incenter of $\triangle PP'P''$.

(3) Let the circumcircle cut II' at L. Then $\angle I'IC = \angle IAC + \angle ACI = \angle BAL + \angle ICB = \angle BCL + \angle ICB = \angle ICL$. Hence, since $\angle ICI'$ is a right angle, L is the midpoint of II'; and O is the midpoint of IJ. Hence, JI' is parallel to OL. But as $\angle BAL = \angle LAC$, arc BL = arc LC. Hence,

$$OL \perp BC \text{ and } \therefore \; JI' \perp BC.$$

Similarly, $JI'' \perp CA$ and $JI''' \perp AB$.

(4) Arc $P''B$ = arc $P'C$. Hence, $\angle P''PB = \angle P'PC$; hence $\angle BPR = \angle RPC$. Hence, $BR : RC = BP : PC$. Similarly, $CR' : R'A = CP' : P'A$, and $AR'' : R''B = AP'' : P''B$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = \frac{BP \cdot CP' \cdot AP''}{PC \cdot P'A \cdot P''B}.$$

But $BP = AP'$, being opposite sides of a rectangle. Similarly, $CP' = BP''$ and $AP'' = CP$. Hence,

$$\frac{BR \cdot CR' \cdot AR''}{RC \cdot R'A \cdot R''B} = 1,$$

numerically. Hence, AR, BR', CR'' are concurrent.

Also solved by A. M. HARDING.

CALCULUS.

373. Proposed by C. N. SCHMALL, New York City.

In the *Encyclopaedia Britannica* article on "Capillary Action" (Vol. 5, p. 268, 11th ed.) it is shown that $1/R_1 + 1/R_2 = p/T$, in the case of a soap bubble, where R_1, R_2 are the *principal radii of curvature* at any point of the bubble; p, the difference of air-pressure; T, the energy per unit area of the film. Employing the principle that the soap bubble tends to assume a form such that the area of its surface is a *minimum* for a *given volume* of air, show by the calculus of variations that $1/R_1 + 1/R_2 = k$, a constant.

SOLUTION BY THE PROPOSER.

We have here to determine the solid which, with a given volume (capacity), contains the least surface. Hence, we have to make the surface integral

$$U = \int\int \sqrt{1 + p^2 + q^2}\, dxdy \tag{1}$$

a minimum, subject to the condition that the volume integral

$$I = \int\int z\, dxdy \tag{2}$$

is constant.

It should be remembered that in this discussion $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, and $t = \partial^2 z/\partial y^2$.

Let k be a constant. Then it is evident that, when the surface (1) is a minimum, the binomial

$$\int\int \sqrt{1+p^2+q^2}\, dxdy + \int\int kz\, dxdy \equiv \int\int (\sqrt{1+p^2+q^2}+kz)dxdy \equiv \int\int V\, dxdy \tag{3}$$

will also be a minimum.

Now, taking x and y as the independent variables, the condition for a mini-

mum is, by the calculus of variations,

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = L, \tag{4}$$

(TODHUNTER'S *Int. Cal.*, p. 374, 5th ed.)

where

$$M = \frac{\partial V}{\partial p} = \frac{p}{\sqrt{1 + p^2 + q^2}},$$

$$N = \frac{\partial V}{\partial q} = \frac{q}{\sqrt{1 + p^2 + q^2}},$$

$$L = \frac{\partial V}{\partial z} = k.$$

(TODHUNTER'S, *Int. Cal.*, p. 372.)

Hence, equation (4) becomes

$$\frac{\partial}{\partial x}\left(\frac{p}{\sqrt{1 + p^2 + q^2}}\right) + \frac{\partial}{\partial y}\left(\frac{q}{\sqrt{1 + p^2 + q^2}}\right) = k \tag{5}$$

or,

$$\frac{r(1 + p^2 + q^2) - (pr + qs)p + t(1 + p^2 + q^2) - (ps + qt)q}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k,$$

or,

$$\frac{(1 + q^2)r - 2pqs + (1 + p^2)t}{(1 + p^2 + q^2)^{\frac{3}{2}}} = k, \tag{6}$$

which is the partial differential equation of the required minimal surface, the integral of which will represent the surface itself.

Again, R_1 and R_2 are known as the *principal radii of curvature* at any point of the surface. The equation giving these is

$$(rt - s^2)R^2 - \sqrt{1 + p^2 + q^2}\,[(1 + p^2)t - 2pqs + (1 + q^2)r]R + (1 + p^2 + q^2)^2 = 0. \tag{7}$$

(GOURSAT-HEDRICK'S *Math. Anal.*, Vol. 1, p. 504, Eq. 13.)

If R_1, R_2, be the roots of this quadratic in R, we have

$$R_1 + R_2 = \frac{\sqrt{1 + p^2 + q^2}\,[(1 + p^2)t - 2pqs + (1 + q^2)r]}{rt - s^2}, \tag{8}$$

$$R_1 R_2 = \frac{(1 + p^2 + q^2)^2}{rt - s^2}. \tag{9}$$

Dividing (8) by (9), we obtain

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{(1 + p^2)t - 2pqs + (1 + q^2)r}{(1 + p^2 + q^2)^{\frac{3}{2}}}. \tag{10}$$

(See EISENHART'S *Diff. Geom.*, p. 126, ex. 3.)

Comparing equations (6) and (10) we have the required result,

$$\frac{1}{R_1} + \frac{1}{R_2} = k.$$

Note 1.—The sphere,

$$x^2 + y^2 + z^2 = a^2 \tag{11}$$

and the cylinder,

$$y^2 + z^2 = b^2, \tag{12}$$

are examples of minimal surfaces satisfying Eq. (6).

Thus, in (11),

$$p = -x/z, \quad q = -y/z, \quad \sqrt{1+p^2+q^2} = a/z,$$

$$\therefore \quad M = -x/a, \quad N = -y/a,$$

and (11) becomes $k + 2/a = 0$; and (12), $k + 1/b = 0$.

Note 2.—In the foregoing solution I have utilized the notation employed in the chapter on the Calculus of Variations in TODHUNTER'S *Integral Calculus*, fifth edition.

374. Proposed by C. N. SCHMALL, New York City.

Show that, on a *Mercator's Chart*, a great circle of a sphere of radius r_1 will be represented by a curve whose equation is of the form

$$c(e^{y/r} - e^{-(y/r)}) = 2 \sin\left(\frac{x}{r} + \theta\right).$$

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

If the latitude and longitude on the above sphere be denoted by the letters φ and θ respectively, θ varying from 0° to 360°, and φ from $-$ 90° to $+$ 90°; then if axes be taken with origin at the center of the sphere with the xy-plane as the plane of the equator, and if longitude be measured from the x-axis, we have for any point on the sphere

$$x = r \cos\varphi \cos\theta, \; y = r \cos\varphi \sin\theta, \; z = r \sin\varphi.$$

The equation of a great circle is obtained by substituting these values in the equation of any diametral plane, $Ax + By + Cz = 0$, and is

$$A \cos\varphi \cos\theta + B \cos\varphi \sin\theta + C \sin\varphi = 0. \tag{1}$$

The sphere is mapped on a *Mercator's Chart* by taking a cylinder tangent to the sphere along the equator and projecting a meridian (θ = const.) on a generating line of the cylinder.

Any point on the sphere on this meridian has for its image on the chart a point on the corresponding generating line at a distance $r \log \tan(\pi/4 + \varphi/2)$ from the equator.

When we develop the cylinder on a plane, we can choose axes in that plane so that the coördinates of this point are

$$x = r\theta, \qquad y = r \log \tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right).$$

Solving these equations for θ and φ, $\theta = x/r$, $\varphi = 2 \arctan(e^{y/r}) - \pi/2$.

Substituting these values in (1), we obtain

$$A \cos\frac{x}{r} + B \sin\frac{x}{r} + C \tan\left\{2 \arctan e^{y/r} - \frac{\pi}{2}\right\} = 0,$$

which reduces at once to the form given, if we let $c = -C/\sqrt{A^2+B^2}$, and $\sin\theta = A/\sqrt{A^2+B^2}$.

II. Solution by A. M. Harding, University of Arkansas.

The equation of the sphere is

$$\frac{x}{r} = \cos u \cos v, \qquad \frac{y}{r} = \cos u \sin v, \qquad \frac{z}{r} = \sin u.$$

Any plane through the center will be given by $ax + by + cz = 0$, or

$$a \cos u \cos v + b \cos u \sin v + c \sin u = 0.$$

Dividing by $\sqrt{a^2 + b^2}$, we obtain $\sin (v + \theta) + (c/\sqrt{a^2 + b^2}) \tan u = 0$, where θ is defined by the equations; $\sin \theta = a/\sqrt{a^2 + b^2}$, $\cos \theta = b/\sqrt{a^2 + b^2}$.

The transformation is

$$\frac{y}{r} = \operatorname{logtan}\left(\frac{u}{2} + \frac{\pi}{4}\right), \qquad \frac{x}{r} = v.$$

Solving the first of these equations for $\tan u$, we obtain

$$\tan u = \frac{e^{y/r} - e^{-y/r}}{2}.$$

Hence, by substitution, we have

$$-\frac{c}{\sqrt{a^2 + b^2}}(e^{y/r} - e^{-y/r}) = 2 \sin\left(\frac{x}{r} + \theta\right),$$

which is of the required form.

Also solved by Paul Capron and the Proposer.

MECHANICS.

278. Proposed by A. M. HARDING, University of Arkansas.

A spherical shell of mass m explodes when moving with negligible velocity at a height of h feet above the ground. The shell is divided into very small particles, each of which moves, after the explosion, away from the center of the shell with a speed v, and ultimately falls to the ground. Find the total mass of the fragments which will be found per unit area at any specified distance from the point vertically underneath the shell.

Solution by H. S. Uhler, Yale University.

Let θ and φ denote the angles which the two trajectories, passing through the same point on the ground, make at the instant of the explosion, with the negative and positive directions of the axis of h respectively. φ must be acute but θ may be obtuse. The familiar equation $x = vt + \frac{1}{2}at^2$ leads to

$$r = \frac{v \sin \theta}{g}(+\sqrt{v^2 \cos^2 \theta + 2gh} - v \cos \theta), \tag{1}$$

$$r = \frac{v \sin \varphi}{g}(+\sqrt{v^2 \cos^2 \varphi + 2gh} + v \cos \varphi). \tag{2}$$

Rationalization of (1) and (2) gives

$$gr^2 + 2rv^2 \sin \theta \cos \theta - 2hv^2 \sin^2 \theta = 0, \tag{3}$$

$$gr^2 - 2rv^2 \sin \varphi \cos \varphi - 2hv^2 \sin^2 \varphi = 0. \tag{4}$$

Letting $\tau \equiv \tan\theta$ and $\gamma \equiv \cot\varphi$, (3) and (4) may be transformed to

$$(gr^2 - 2hv^2)\tau^2 + 2rv^2\tau + gr^2 = 0,$$

$$gr^2\gamma^2 - 2rv^2\gamma - (2hv^2 - gr^2) = 0.$$

Hence, under the conditions of the problem

$$\tau = \frac{r(v^2 + R)}{2hv^2 - gr^2}, \qquad \gamma = \frac{v^2 + R}{gr},$$

where

$$R \equiv + \sqrt{v^4 + 2ghv^2 - g^2r^2}.$$

Consequently

$$\cos\theta = \frac{2hv^2 - gr^2}{v\sqrt{2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)}}, \tag{5}$$

$$\cos\varphi = \frac{v^2 + R}{v\sqrt{2(gh + v^2 + R)}}, \tag{6}$$

By considering solid angles, it may be readily seen that the mass which is deposited on the circular area of radius r, corresponding to the cone of half-angle θ, is given by $\frac{1}{2}m(1 - \cos\theta) \equiv m_\theta$. Similarly the mass issuing from the upper or φ cone and falling on the same area equals $\frac{1}{2}m(1 - \cos\varphi) \equiv m_\phi$. The total mass on the area πr^2 is, therefore, $m_\theta + m_\phi \equiv m_r$. The total mass distributed over the ring $2\pi r dr$ is $(dm_r/dr)dr$, so that the surface density is given by

$$\sigma_r = \frac{1}{2\pi r}\left(\frac{dm_\theta}{dr} + \frac{dm_\phi}{dr}\right). \tag{7}$$

Assuming $v^4 + 2ghv^2 - g^2r^2 \neq 0$ the derivatives involved in (7) may be formed by the aid of (5) and (6), so that, in general

$$\sigma_r = \frac{m}{4\pi vR}\left\{\frac{(gr^2 + 2hR)(v^2 + R)^2}{[2(2h^2v^2 - ghr^2 + r^2v^2 + r^2R)]^{\frac{3}{2}}} + \frac{g^2(2gh + v^2 + R)}{[2(gh + v^2 + R)]^{\frac{3}{2}}}\right\}.$$

Special cases:

(*a*) When $R = 0$, that is, when r attains its superior limit $v/g\sqrt{v^2 + 2gh}$ the θ and φ trajectories coincide, the angles θ and φ become supplementary, and $\varphi = \tan^{-1}(1/v\sqrt{v^2 + 2gh})$.

(*b*) Taking $g = 32$ ft./sec.^{2}, $h = 5{,}000$ ft., $v = 1{,}024$ ft./sec., the maximum value of r equals $4{,}608\sqrt{66} \doteq 37{,}435.57$ ft. For $r = 0$, $\sigma = 3{,}669 \times 10^{-12}$m. When $2hv^2 - gr^2 = 0$ (*i. e.*, $\theta = \pi/2$) $\sigma = 1{,}612 \times 10^{-13}$ m. When $R = 6{,}604$, $\sigma = 1{,}823 \times 10^{-11}$ m. In the last instance $r \doteq 37{,}435.0001$ ft. If the entire mass were distributed uniformly over the maximum circle the surface density would be approximately $2{,}271 \times 10^{-13}$ m.

MECHANICS.

296. Proposed by C. N. SCHMALL, New York City.

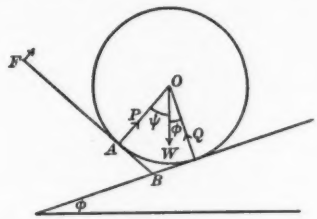
A force F is exerted in moving a horizontal cylinder up an inclined plane by means of a crowbar of length l. If R be the radius of the cylinder, W its weight, φ the inclination of the plane to the horizontal and ψ the inclination of the crowbar to the horizon, show that

$$F = \frac{WR\sin\varphi}{l[1 + \cos(\varphi + \psi)]}.$$

SOLUTION BY A. M. HARDING, University of Arkansas.

The cylinder is in equilibrium under the three forces P, Q, and W. Hence, by Lami's Theorem, $P : Q : W = \sin\varphi : \sin\psi : \sin(\varphi + \psi)$.

$$\therefore \quad P = \frac{W \sin\varphi}{\sin(\varphi + \psi)}, \qquad Q = \frac{W \sin\psi}{\sin(\varphi + \psi)}.$$



From the triangle AOB we obtain

$$AB = R \tan\frac{\varphi + \psi}{2} = R\frac{\sin(\varphi + \psi)}{1 + \cos(\varphi + \psi)}.$$

Taking moments about B we obtain

$$F \times l = P \times AB = \frac{WR \sin\varphi}{1 + \cos(\varphi + \psi)};$$

whence

$$F = \frac{WR \sin\phi}{l[1 + \cos(\phi + \psi)]}.$$

Solved similarly by CLIFFORD N. MILLS.

NUMBER THEORY.

210. Proposed by ELMER SCHUYLER, Brooklyn, N. Y.

If a and b are relatively prime and $(a + b)$ is even, then $(a - b)ab(a + b) \equiv 0$, (mod. 24).

SOLUTION BY R. M. MATHEWS, Riverside, California.

The statements a and b relatively prime and $a + b$ even imply

$$a = 2m + 1, \; b = 2n + 1$$

and

$$a + b \equiv 0 \pmod 2,$$

$$a - b \equiv 0 \pmod 2.$$

If m and n both odd or both even,

$$a - b \equiv 0 \pmod 4.$$

If m and n be one odd and the other even,

$$(a + b) \equiv 0 \pmod 4.$$

Thus, in any case,

$$(a + b)(a - b) \equiv 0 \pmod 8,$$

a and b each belong to one of the types $3k$, $3k + 1$, $3k + 2$.

If either be $3k$, then

$$ab \equiv 0, \text{ mod. } 3.$$

If both be of type $3k + 1$, or $3k + 2$, then

$$a - b \equiv 0, \text{ mod. } 3.$$

If one be of type $3k + 1$ and the other $3k + 2$, then

$$a + b \equiv 0, \text{ mod. } 3.$$

Thus in any case

$$(a + b)ab(a - b) \equiv 0 \pmod{24}.$$

QUESTIONS AND DISCUSSIONS.

EDITED BY U. G. MITCHELL, University of Kansas.

NEW QUESTIONS.

29. While studying the problem of two equal rough bodies, connected by an inelastic wire, resting on an inclined plane, Professor Clifford N. Mills of South Dakota State College met the following interesting expression. If $1/a$ and $1/a + 1$ are the coefficients of friction, the tension of the wire when the bodies are about to descend becomes a multiple of $1/a[1/a + 1/a + 1]$. This, when simplified, becomes $(2a + 1)/2a(a + 1)$. If $2a + 1$ and $2a(a + 1)$ represent the base and altitude of a right triangle the hypothenuse is $2a^2 + 2a + 1$. Therefore this gives a series of numbers which satisfy the relation $x^2 + y^2 = z^2$, if a is given any value whatsoever. Professor Mills desires to know if this will give all the integers which satisfy the condition that the sum of the squares of two integers equals the square of an integer.

REPLIES.

23. What should be done with the theory of limits in elementary geometry? Should the recommendation of the National Committee of Fifteen be universally adopted? If not, what better disposition of the subject can be made?

REPLY BY R. M. MATTHEWS, Riverside, California.

In elementary euclidean plane geometry the theory of limits is used at four places. The last of these is for the evaluation of a particular sequence which defines a definite limit π and so is different from the other three which are the theorems fundamental to proportion and the comparison of areas. The first is, "In a circle, central angles are proportional to their intercepted arcs." It depends upon the theorem that equal central angles intercept equal arcs, a relation which is established by congruence. The second is, "A set of parallel lines intercept proportional segments on all transversals." It depends upon the theorem that parallels which intercept equals on any transversal intercept equals on every transversal, a relation also established by congruence. The third is, "Rectangles are proportional to the products of their bases and altitudes"

and its proof is essentially an extension of the theory of measurement implied in the second.

These three propositions can be treated in either of two ways. There is the purely geometric theory of proportion which Euclid developed and which we do not teach. It is not essentially dependent on the idea of ratio or on a complete theory of arithmetic. On the other hand, there is the method of basing geometric proportion on the theory of arithmetic proportion. This latter theory depends on the idea of the ratio of two numbers and connection between arithmetic and geometry is made by defining the ratio of two geometric magnitudes as the ratio of their numerical measures. Accordingly, the whole matter becomes a question of measurement and there would be no difficulty if all magnitudes were "commensurable." This distinction of "commensurable" and "incommensurable" is a survival from Euclid's purely geometric treatment and is needless when a complete system of real numbers is applied to the theory of measurement.

In the measurement of linear segments, the standard distinction is that two segments are commensurable when there exists a segment which can be applied to each an *integral* number of times; otherwise they are incommensurable. Thus, the primary concept is *the number of times one segment is contained in another.* In this primary concept we retain the impression of the simple operation of applying the one segment to the other a step at a time and so argue that "half a time" is absurd. Nevertheless, we practically abandon this primary impression and enlarge our idea of "number of times" when we attach a definite meaning to the statement that segment AB is contained $7\frac{56}{81}$ (for example) times in segment XY. Theoretically we conceive of AB as divided into 81 equal parts and that one of these parts is contained $(7 \times 81 + 56)$ times in XY. The notion of "times contained" has been extended so as to render available all the advantages of the rational number system.

The side and diagonal of a square are declared incommensurable because no segment exists which is contained in both a rational number of times. We need not be confined to rational numbers, however, and the idea of "number of times contained" may be enlarged so as to comprehend every case.

Arithmetically, $\sqrt{2}$ may be defined as the class of all convergent infinite sequences whose limits are the same as the limit of the infinite sequence of rational numbers 1, 1.4, 1.41, 1.414, ...

Geometrically, the side of a square may be applied, in the primary sense, to the diagonal once and a point A_1 determined. A tenth part of the side may be applied 14 times and a point A_2 determined; a hundredth part may be applied 141 times and a point A_3 determined; and so on. This process defines on the diagonal an infinite sequence of points which converges to the end point of the diagonal as a limit. If a different subdivision of the side be used, a different sequence of points would result which, nevertheless, would be equivalent to the first in that it too would converge to the end point of the diagonal.

Thus, on the one hand, there is a class of equivalent convergent sequences of rational numbers, and, on the other hand, a corresponding class of equivalent

convergent sequences of points. Dedekind's axiom is to the effect that to the irrational number defined by the class of numerical sequences corresponds one and only one point, namely, that which is the limit point of the corresponding sequences of points. This, it seems to me, is the adequate enlargement of our idea of "number of times contained" so that we can speak as definitely and rigorously of one segment being contained $\sqrt{2}$ times in another segment as we do when we say that one is contained 1.4 times in another. In brief, *for the theory of measurement in elementary geometry the limit process has been applied once for all in the correlation of real numbers with the points on a line.*

The present difficulties are encountered because we are allowing ourselves to be confined to the number system that Euclid knew—and which was adequate for his method of treating proportion—while trying to treat proportion by arithmetic, which he did not do.

According to the foregoing argument, the results of modern research justify us in regarding every two segments as commensurable and, consequently, all theorems on proportional lines, angles and areas will be proved as they now are in the conventional "commensurable case."

It may be asked, however, what must be taught to high school students for the Dedekind axiom and how a revised proof would run. The usual distinction between commensurable and incommensurable would be replaced by the explicit assumption: *For every two like magnitudes there exists a number that denotes the number of times the one is contained in the other.*

Not only is the equivalent of this axiom at the foundation of analysis but it is the very assumption that seems reasonable to the student. Moreover, we do use it. Let us, therefore, make it explicit.

In secondary school work irrationals are introduced in arithmetic and some use is made of them in the first course in algebra, Since we proceed as if our students had the whole real number system, is it unreasonable to apply it to geometry? Possibly our methods of teaching irrational numbers may need revision, but that is a question for arithmetic and algebra rather than for geometry.

With the foundations for a simple and complete theory of measurement laid in the needed assumption, the proofs for the three important theorems would be simplified. In fact, they would be virtually the same as in the ordinary "commensurable case" except that the numbers used would represent any real numbers instead of merely integers.

The case of limits in the measurement of the circle is different, since in that case the difficulty is not with the limit (whose existence is postulated) but with the setting up of a numerical sequence which will evaluate that limit approximately.

COMMUNICATION.

To the Editor of the Monthly: It seems that a criticism which I made in reviewing Dowling and Turneaure's *Analytic Geometry* in the March number of the Monthly was not very clear, since Professor Dowling states in the May number that the proof which I said was lacking is given on page 58, whereas I

find there no answer to the point which I wished to raise. Inasmuch as the criticism applies also to another recent text (SMITH and GALE: *New Analytic Geometry*) I will ask space for a more detailed statement of my point.

By definition, the equation of the locus of a point satisfying a given condition is an equation in the variables x and y representing coordinates such that, (1) if a point $P(x, y)$ is on the locus it satisfies the equation, and conversely, (2) if a point $P(x, y)$ satisfies the equation it is on the locus.[1]

In deriving the equation of a straight line, Dowling and Turneaure proceed as follows on pages 57, 58:

> Choose any point $P(x, y)$ on the line joining P_1 and P_2 (given points). Then, slope of segment P_1P = slope of segment P_1P_2. But (Art. 11), slope of $P_1P = (y - y_1)/(x - x_1)$, and slope of $P_1P_2 = (y_2 - y_1)/(x_2 - x_1)$. Therefore $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$. (1)

Immediately following this, the authors change this equation into the slope-point form, the slope form, and the intercept form. They then say, "Equations (1), (2), (3), and (4) are all standard forms of the equation of a straight line." My criticism is that they have not shown that any of these equations is the equation of a straight line. They have shown that equation (1) satisfies the first requirement of the definition of the equation of a locus (*i. e.*, if $P(x, y)$ is on the line it satisfies the equation) but they have not proved the "and conversely" part of the definition to be satisfied (*i. e.*, they have not shown that if a point $P(x, y)$ satisfies the equation it lies on the locus). A complete proof of the "and conversely" is as difficult as the part of the proof which they have given, and it should not be omitted.

Following the last statement quoted, the authors name the several standard forms, and then from these equations conclude that, "the equation of a straight line is of first degree in the variables x and y." They then say, "Conversely, it may be shown that any equation of the first degree in the variables x and y is the equation of a straight line." In the proof of this theorem they assume that they have proved that $y = mx + b$ is the equation of a straight line. Obviously then the proof of this "conversely" cannot nullify the criticism I have made above.

Smith and Gale have made the same omission on page 58 of their text.

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[1] This is essentially the way in which the definition is given in both texts. See page 55 of the former and page 32 of the latter.

NOTES AND NEWS.

EDITED BY W. D. CAIRNS, Oberlin College.

Professor R. M. BARTON of Lombard College, Galesburg, Ill., has been made dean and acting president of that institution.

Mr. GEORGE RUTLEDGE and Dr. B. B. LIBBY have been appointed instructors in mathematics in the Massachusetts Institute of Technology.

Dr. RAYMOND B. ROBBINS has been appointed to an instructorship in mathematics in the Sheffield Scientific School of Yale University.

Professor E. D. ROE, Jr., of Syracuse University, and Mr. F. E. CARR, of Oberlin College, spent the summer in astronomical work at Yerkes Observatory.

Miss ANNA D. LEWIS, instructor in mathematics and astronomy at Mt. Holyoke College, has been appointed head of the department of mathematics in Kentucky College for Women.

Henry Holt and Company has announced RIETZ, CRATHORNE and TAYLOR'S "School Algebra," also YOUNG and SCHWARTZ'S "Elementary Geometry."

The January–February number of the *Rendiconti del Circolo Matematico di Palermo* contains a biographical account of the work of Professor G. B. Guccia by Professor M. DE FRANCHIS.

The June number of the *Pahasapa Quarterly* of the South Dakota School of Mines pays a tribute to Professor H. L. MCLAURY in recognition of his completion of twenty years as head of the department of mathematics.

Professor ELLERY W. DAVIS, of the University of Nebraska, gave the annual address before the Iowa Academy of Sciences on April 30 on the subject "Uncertainties."

The Open Court for August, 1915, contains an article by Professor F. CAJORI on "The Life of William Oughtred," who contributed to the propagation of mathematical knowledge in England during the early part of the seventeenth century.

Professor C. J. KEYSER, of Columbia University, delivered the commencement address at the University of Oregon in June. He gave also the annual address before the chapter of Sigma Xi of the University of Washington on "Science and Religion, or the Rational and the Super-rational."

In the June number of *School Science and Mathematics* Professor M. O. TRIPP, of Olivet College, enumerates "Some simple applications of elementary algebra to arithmetic," and Mr. H. C. Wright describes the use of mathematical apparatus in the university high school of the University of Chicago.

Mr. W. L. HART will be instructor in mathematics and astronomy at the University of Montana for the first semester of the present year, in place of Professor L. S. Hill, who has accepted an appointment at Princeton University. Mr. Hart will return to Chicago after the holidays to finish his work for the doctorate at the University of Chicago.

Miss OLIVE HAZLETT, who has just finished her work for the doctorate at the University of Chicago, has returned to her home in Boston. She holds a traveling scholarship from the Association of Collegiate Alumnæ, which she had hoped to use in Europe during the coming year, but war conditions may make this impracticable.

An article of interest to teachers of mathematics was published in *School and Society* for May 29, under the title "College Pedagogy," by LOUIS W. ROPER, of Pennsylvania State College. A list is given of 33 qualities of a good college instructor made by twenty college juniors and seniors, each of whom had come in contact with about forty instructors in high school and college.

Dr. S. W. REAVES, professor of mathematics at the University of Oklahoma, is resuming his work after a year spent on sabbatic leave studying in Chicago. He completed work for his doctorate at the close of the summer quarter.

Science for July 2 prints the Halley Lecture for the year 1915 on "Measurements of the distances of the stars," in which the Astronomer Royal, Sir F. W. Dyson, recounts the progress made up to the present in measuring the parallaxes of the nearer fixed stars.

Professor ARTHUR D. PITCHER, of Dartmouth College, has been appointed head of the department of mathematics in Adelbert College, Western Reserve University; and Dr. JOHN M. STETSON has been appointed instructor in mathematics.

The *Mathematical Gazette* for May contains two papers of special interest: "Notes on the Board of Education circular No. 851" (a revision of an earlier circular on the teaching of geometry), by Miss M. J. PARKER, and the first instalment of "The discovery of logarithms by Napier," by Professor H. S. CARSLAW. It will be interesting to compare the latter as it appears with Professor Cajori's "History of the exponential and logarithmic concepts" which appeared in the MONTHLY during the year 1913.

An organization of men who are teaching mathematics in secondary schools of Chicago and vicinity has just started on its second year of activity. They meet monthly for an informal dinner and discussion of live topics of the day in mathematics.

In the thirteenth yearbook of the National Society for the Study of Education appears a paper by H. C. Morrison, superintendent of public instruction for New Hampshire, under the title "Reconstructed mathematics in the high school: the adaptation of instruction to the needs, interests, and capacities of students." This discussion, guided largely by the utilitarian viewpoint, examines critically the extent to which arithmetic, geometry and algebra are needed by different groups of students, *e. g.*, domestic arts, agriculture, mechanic arts, etc., and insists that each shall have its own specially organized mathematics, in charge of specialists in that curriculum.

Mr. W. D. REEVE, formerly instructor in mathematics at the University of Chicago High School, is now in charge of the mathematical courses in the department of education in the University of Minnesota. Mr. Reeve is a graduate of the University of Chicago and has been on the High School staff for several years.

In the last volume of the "Proceedings of the Society for the Promotion of Engineering Education" more than 50 pages are devoted to articles on the teaching of college mathematics and discussion of this subject. The articles are: "Practical mathematics," by Professors FRANKLIN, MACNUTT, and CHARLES; "The proper use of the differential in calculus," by Professor E. V. HUNTINGTON, and "The calculus without symbols," by Professor E. R. HEDRICK. Professor Huntington brings out very clearly, by quotations from texts in calculus, how vague is the presentation of so simple and fundamental concept as that of the differential. The spirited discussion of the first two articles will be found edifying in spots and quite entertaining throughout. No teacher of calculus can afford to miss reading these papers and their discussion.

After the present year the College Entrance Examination Board will conduct all entrance examinations for Yale, Princeton, and Harvard Universities. In view of this greatly increased scope of the Board's work, it is anticipated that 10,000 candidates will be dealt with in the June, 1916, examinations. The readers in mathematics for the present year are as follows:

Algebra.—Professor C. R. MacInnes (chief reader), Princeton; Edward B. Chamberlain, Franklin School, New York; Miss E. B. Cowley, Vassar; C. A. Ewing, Tome School, Port Deposit; Miss Marcia Latham, Hunter; Professor W. R. Longley, Yale; I. F. McCormick, Albany Academy; George W. Mullins, Barnard; C. B. Walsh, Ethical Culture School, New York.

Geometry.—Professor Virgil Snyder (chief reader), Cornell; F. C. Baldy, St. Mark's School, Southboro; G. M. Conwell, Yale; Professor H. N. Davis, Harvard; L. S. Dederick, Princeton; J. G. Estill, Hotchkiss School, Lakeville; J. R. Gardner, Allen-Stevenson School, New York; G. M. Green, Harvard; Professor J. C. Hardy, Williams; Lester E. Lynde, Phillips Academy, Andover; F. J. McMackin, Columbia; E. B. Morrow, Gilman Country School, Roland Park; Professor A. D. Pitcher, Western Reserve; H. W. Reddick, Columbia; H. L. Sweet, Phillips Exeter Academy; L. L. Whitney, Collegiate School, New York.

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THE TEACHING OF MATHEMATICS.

By H. E. SLAUGHT.

Owing to a wide divergence of opinion among college and university men as to the proper interpretation of the term *pedagogy*, and especially in view of the varying notions of that term in the minds of mathematicians, it seems desirable to formulate as definitely as possible what is conceived to be the relation of the MONTHLY to the question of *teaching* in the field of collegiate mathematics.

A certain kind of formal didactics, which at one time prevailed widely in schools for the training of elementary teachers, and which also characterized the earlier attempts at pedagogical training for secondary teachers, has served to alarm many college and university men, whenever the question of college teaching is mentioned, lest the same stilted and formal methods be carried over into the collegiate field. These men have seen some horrible example of a college teacher using what they describe with fine scorn as "high school methods," and thereby sacrificing the higher spirit of the subject to the lower criterion of *so-called pedagogical form*. They say, and with good reason let it be granted, that they would far rather have a young college instructor with good training, fine spirit, and high ideals, who had no teaching experience and no pedagogical training, rather than one whose initiative and individuality is so hedged about with formal rules on teaching and methods of presentation that he becomes a mere machine. And having thus spoken, these people feel that they have said the last word on the teaching of mathematics so far, at least, as the collegiate field is concerned. They think that the teacher is born, not made, that if the young fellow has it in him he will become a good teacher after a certain period of floundering, and if it is not in him, no amount of pedagogical training can make him a good teacher.

There is so large an element of truth in the foregoing argument that many are fully convinced by it. It is true that a bright student from the graduate school may go directly to the teaching of college freshmen and may eventually become proficient at the work, just as it is also true that a student may go directly from the medical school to the independent practice of medicine without the professional training of the clinic and the hospital interneship.